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Finite strain—beam theory

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Abstract

An appropriate strain energy density for an isotropic hyperelastic Hookean material is proposed for finite strain from which a constitutive relationship is derived and applied to problems involving beam theory approximations. The physical Lagrangian stress normal to the surfaces of a element in the deformed state is a function of the normal component of stretch while the shear is a function of the shear component of stretch. This paper attempts to make a contribution to the controversy about who is correct, Engesser or Haringx with regard to the buckling formula for a linear elastic straight prismatic column with Timoshenko beam-type shear deformations. The derived buckling formula for a straight prismatic column including shear and axial deformations agrees with Haringx's formula. Elastica-type equations are also derived for a three-dimensional Timoshenko beam with warping excluded. When the formulation is applied to the problem of pure torsion of a cylinder no second-order axial shortening associated with the Wagner effect is predicted which differs from conventional beam theory. When warping is included, axial shortening is predicted but the formula differs from conventional beam theory.

Keywords: Finite strain; Elastica; Elasticity; Beam theory; Column buckling; Wagner effect; Torsion

1. Introduction

The Euler buckling load for a straight prismatic column was first modified by Engesser (1889, 1891) to include the effects of shear deformations. The solution to this problem is used for the design of helical springs, elastometric bearings, sandwich plates, built-up and laced columns, and composite materials (see Bazant, 1971, 2003; Kardomateas and Dancila, 1997; Bazant and Cedolin, 1991; Gjelsvik, 1991; Simo et al., 1984a; Simo and Kelly, 1984; Timoshenko and Gere, 1963). Engesser's solution predicts a limit on the buckling load equal to the shear stiffness as the slenderness ratio becomes very small. This was not confirmed by experiments on very short highly compressed helical springs which showed that very short springs do not buckle. Haringx (1942) developed an alternate buckling formula which predicted an infinite buckling load as the slenderness approached zero. His formula agreed well with experimental results for short

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springs. Recently Haringx's buckling formula was also shown to provide better predictions for sandwich columns (see Bazant, 2003).

There have been several authors who have discussed the merits of the two approaches (see Timoshenko and Gere, 1963; Bazant, 1971; Zielger, 1982; Reissner, 1972, 1982; Bazant and Cedolin, 1991; Gjelsvik, 1991). The arguments for and against were debated in two papers in 1982, one by Zielger (1982) who supported Engesser approach and Reissner (1982) who supported Haringx's approach. Both approaches used the constitutive law $Q = GA\varphi$ where Q is the shear force, G is the shear modulus, A is the cross-sectional area and φ is the shear angle. Both, however, assumed a different orientation for Q and the axial force N on the cross-section, as shown in Fig. 1. Engesser assumed the axial force to be tangential to the centroidal axis of the beam and the shear force to be perpendicular to this, while Reissner assumed that the axial force was normal to the cross-section and the shear force perpendicular and within the plane of the cross-section. The two approaches yield different formulas for the buckling load. A third approach which is based on assuming a Hookean constitutive law between the Green's strain tensor and the second Piola–Kirchhoff stress tensor yields results which are identical to Engesser's. In this third approach, the actions/stresses consisted of an axial force which is tangential to the centroidal axis of the beam and a shear force which lies within the plane of the cross-section as shown in Fig. 1. Again $Q = GA\varphi$ is assumed.

Bazant (1971, 2003) considered several finite strain formulations each of which assumed a Hookean stress–strain relationship with identical elastic constants. Each formulation predicted different buckling loads. Bazant (1971) and Bazant and Cedolin (1991) was able to show that the Engesser formula resulted if one used a formulation based on the Green's strain tensor and the second Piola–Kirchhoff stress tensor. Haringx's formula resulted if the Doyle–Ericksen strain measure with $m = -2$ was used. This strain measure is identified as the contravariant Almansi strain tensor (refer to Ogden, 1997, p. 119). Bazant (1971, 2003) concluded that all finite strain "formulations are equivalent because the tangential elastic moduli of the material cannot be taken the same but rather must have different values in each formulation". One needed, however, to establish a reference Hookean conjugate strain–stress formulation from which all others could be transformed.

Bazant (1971) and Bazant and Cedolin (1991) also showed that Haringx's equation could be obtained from Engesser's formula if the shear modulus in Engesser's formula was replaced by $G - N/A$ where N is

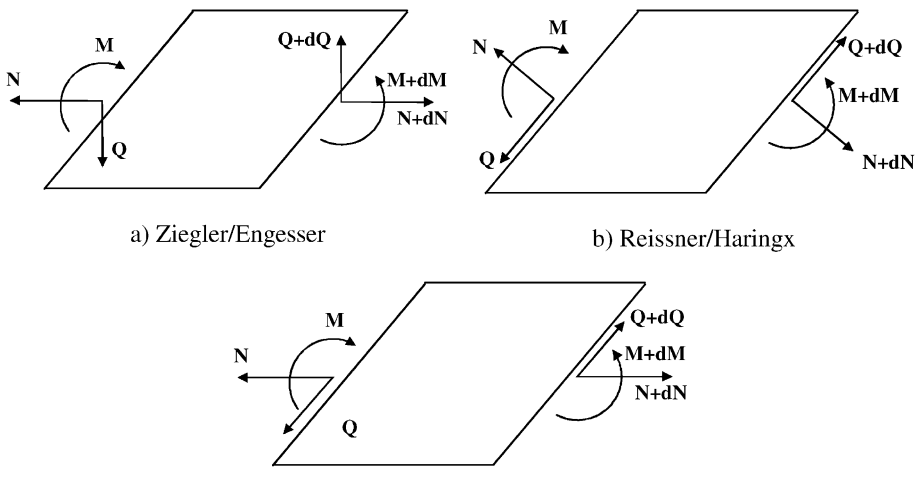


Fig. 1. Two-dimensional beam actions.

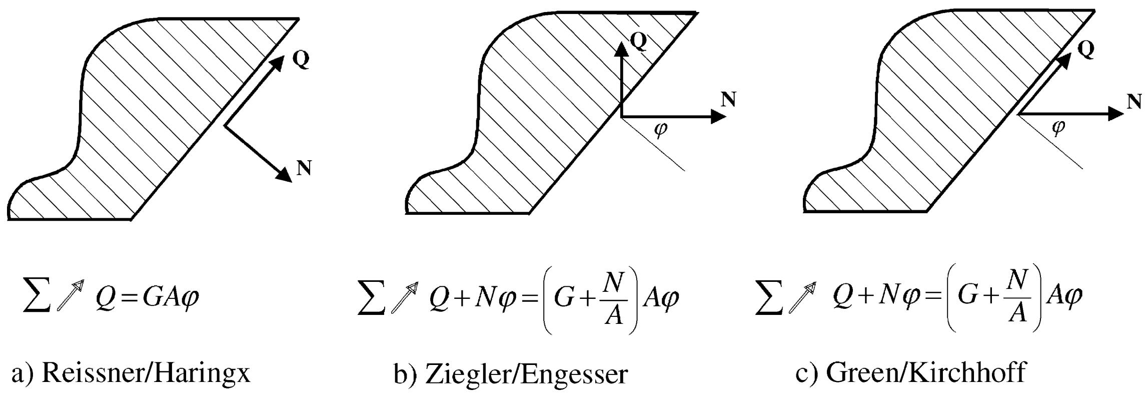


Fig. 2. Shear equilibrium.

the applied axial force. A similar conclusion was reached by Simo et al. (1984a) who used a Green's strain tensor-second Piola–Kirchhoff stress formulation to study the buckling of a beam flexible in shear. This can also be seen by examining equilibrium in the direction of the plane of the cross-section as shown in Fig. 2 and assuming small deflections. The three approaches yield the same result if the G used for the Engesser and Green/Kirchhoff approaches is replaced by $G - N/A$.

In a previous paper (Attard, 2003), a strain energy density was proposed for finite strain isotropic hyperelastic materials. This strain energy density is used to derive constitutive relationships for problems involving beam theory. The buckling formula for a straight prismatic column including shear and axial deformations is derived and agrees with Haringx's formula. Elastica-type equations are derived for a three-dimensional Timoshenko beam with warping excluded. The example of pure torsion of a cylinder is examined as the proposed formulation predicts no second-order axial shortening under pure torsion. This differs from conventional finite strain predictions associated with the Wagner effect. It is further shown that if one includes warping then under pure torsion axial shortening is predicted.

2. Strain energy density for an isotropic Hookean material

A simple non-negative strain energy density for a compressible isotropic Hookean material is (see Attard, 2003):

$$\mathrm{d}U = \tfrac{1}{2}G(I_\lambda - 3) + \tfrac{1}{2}\Lambda(\ln J)^2 - G\ln J\,\mathrm{d}V \tag{1}$$

where the material constants are defined by

$$G = \frac{E}{2(1+\mu)} \qquad \Lambda = \frac{E\mu}{(1+\mu)(1-2\mu)} \tag{2}$$

with E being the elastic modulus, G the shear modulus, μ the Poisson's ratio and Λ the Lamé constant. The invariants I_λ and J are defined by:

$$I_\lambda = g^{ij}\hat{g}_{ij} = (\lambda_{\mathrm{p}1})^2 + (\lambda_{\mathrm{p}2})^2 + (\lambda_{\mathrm{p}3})^2 = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 \tag{3}$$

$$J = (III_\lambda)^{1/2} = \sqrt{\frac{\det(\hat{g}_{ij})}{\det(g_{ij})}} = \frac{\sqrt{\hat{g}}}{\sqrt{g}} = \lambda_{p1}\lambda_{p2}\lambda_{p3}$$
$$= \lambda_1\lambda_2\lambda_3(1 + 2\cos\hat{\phi}_{12}\cos\hat{\phi}_{13}\cos\hat{\phi}_{23} - \cos^2\hat{\phi}_{12} - \cos^2\hat{\phi}_{13} - \cos^2\hat{\phi}_{23})^{1/2} \quad (4)$$

where g^{ij} is the contravariant metric tensor in the undeformed state, $\hat{g}_{ij}$ is the covariant metric tensor in the deformed state, λ_{p1}, λ_{p2} and λ_{p3} are the principal stretches, λ_1, λ_2 and λ_3 are the stretches when the initial coordinate system is Cartesian and $\hat{\phi}_{ij}$ are the angles between the ith and jth tangent base vectors in the deformed state. Eq. (1) can be approximated to the order of $O([J-1]^3)$ as follows.

$$dU = \tfrac{1}{2}G(I_\lambda - 3) + \tfrac{1}{2}\Lambda(\ln J)^2 - G\ln J\,dV = \tfrac{1}{2}G(I_\lambda - J^2 - 2) + \tfrac{1}{2}(\Lambda + 2G)(J-1)^2 + O([J-1]^3)\,dV \quad (5)$$

The strain energy density defined in Eq. (5) can also be expressed in the following form:

$$dU = \tfrac{1}{2}G(I_\lambda - J^2 - 2) + \tfrac{1}{2}(\Lambda + 2G)(J-1)^2 + O([J-1]^3)\,dV$$
$$= \tfrac{1}{2}G([\lambda_{n1}^2 + \lambda_{s1}^2] + \lambda_2^2 + \lambda_3^2 - \lambda_{n1}^2[\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}] - 2) + \tfrac{1}{2}(\Lambda + 2G)(\lambda_{n1}[\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}]^{1/2} - 1)^2$$
$$+ O([J-1]^3)\,dV \quad (6)$$

where $J/\lambda_{n1} = [\hat{g}_{22}\hat{g}_{33} - \hat{g}_{23}\hat{g}_{23}]^{1/2}$ is the ratio of the surface area bound by $\hat{\mathbf{g}}_2\,d\theta^2$ and $\hat{\mathbf{g}}_3\,d\theta^3$ in the deformed state to the initial state and $\lambda_{n1} = 1/\sqrt{\hat{g}^{11}}$ and $\lambda_{s1} = \sqrt{\hat{g}_{11} - (1/\sqrt{\hat{g}^{11}})}$ are the normal and tangential components of the stretch λ_1, respectively, as shown in Fig. 3. One level of beam theory approximation is to assume that the beam cross-sectional shape and area remains unchanged during deformation and that the stress state is essentially uniaxial. If one assumes $J/\lambda_{n1} = 1$ then Eq. (6) becomes:

$$dU \cong \tfrac{1}{2}G(I_\lambda - J^2 - 2) + \tfrac{1}{2}(\Lambda + 2G)(J-1)^2 + O([J-1]^3)\,dV$$
$$\cong \tfrac{1}{2}G((\lambda_{s1})^2 + \lambda_2^2 + \lambda_3^2 - 2) + \tfrac{1}{2}(\Lambda + 2G)(\lambda_{n1} - 1)^2 + O([\lambda_{n1} - 1]^3)\,dV \quad (7)$$

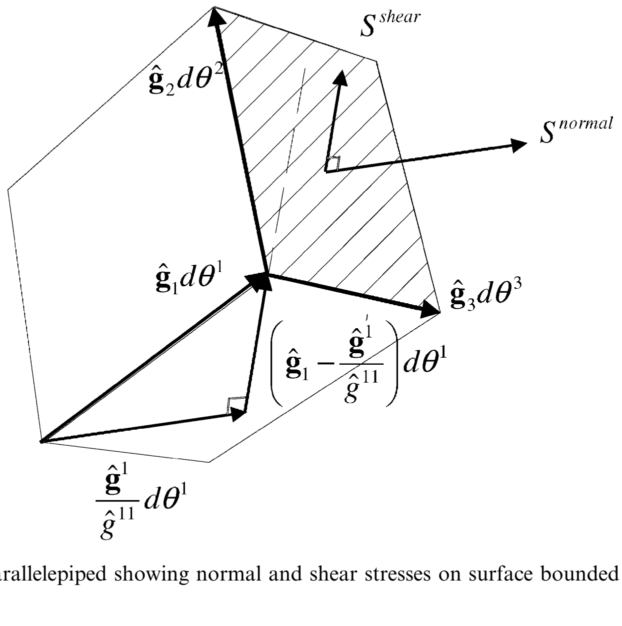


Fig. 3. Deformed parallelepiped showing normal and shear stresses on surface bounded by $\hat{\mathbf{g}}_2 d\theta^2$ and $\hat{\mathbf{g}}_3 d\theta^3$.

The second term in Eq. (7) is the strain energy due to extension normal to the deformed surface (in the direction $\hat{\mathbf{g}}^1$). The material parameter governing this term is $\Lambda + 2G$ and not the elastic modulus E as would be expected for a uniaxial stress state. This is because restraining the cross-section shape involves the application of lateral stresses which would not be present under a uniaxial stress state. A further approximation in beam theory is to replace $\Lambda + 2G$ by E in Eq. (7). To distinguish when this approximation is made in this paper, $\Lambda + 2G$ will be replaced by E^*. Eq. (7) then becomes:

$$\mathrm{d}U \cong \tfrac{1}{2}G((\lambda_{s1})^2 + \lambda_2^2 + \lambda_3^2 - 2) + \tfrac{1}{2}E^*(\lambda_{n1} - 1)^2 + \mathrm{O}([\lambda_{n1} - 1]^3)\,\mathrm{d}V \tag{8}$$

The constitutive law for the physical Lagrangian stresses normal and tangential to the beam cross-section derived from Eq. (8) are (refer to Attard, 2003):

$$\begin{aligned} s_{\mathrm{R}}^{\mathrm{normal}} &= \frac{\partial \mathrm{d}U}{\partial \lambda_{n1}} \cong E^*(\lambda_{n1} - 1) \\ s_{\mathrm{R}}^{\mathrm{shear}} &= \frac{\partial \mathrm{d}U}{\partial \lambda_{s1}} = G\lambda_{s1} \end{aligned} \tag{9}$$

Note this assumes that λ_1 or λ_2 are not functions of λ_{n1}. The subscript 'R' used in the above notation of stresses is to indicate that these stresses are in agreement with Reissner's proposal for beam actions (refer to Fig. 1b). The shear component is in the direction defined by the vector $\hat{\mathbf{g}}_1 - (\hat{\mathbf{g}}^1/\hat{g}^{11}) = -(\hat{g}^{12}/\hat{g}^{11})\hat{\mathbf{g}}_2 - (\hat{g}^{13}/\hat{g}^{11})\hat{\mathbf{g}}_3$ (see Attard, 2003).

3. Uniform uniaxial tension/compression

The first example considered here is the analysis of a prismatic bar under uniform axial tension or compression. The cross-sectional shape is assumed to undergo no lateral strains within the cross-sectional plane and therefore $\lambda_2 = 1$, $\lambda_3 = 1$ and $\hat{g}_{23} = 0$. Equilibrium will only be satisfied in the longitudinal direction as the maintenance of the cross-sectional shape implies the presence of lateral stresses to contra the effects of volume dilation.

The initial axis system is taken as a Cartesian rectangular system. The longitudinal axis of centroids of the undeformed beam is taken as the x- or 1-axis. The orthogonal axes perpendicular to the x-axis are taken as the z- or 2-axis and y- or 3-axis. Under uniaxial deformation it is assumed that all points within the cross-section undergo a uniform longitudinal displacement $u(x)$. Second and higher derivatives of $u(x)$ are assumed to vanish. All displacement functions in this paper are taken as Lagrangian. The position vector $\widehat{\mathbf{R}}$ for a particle that initially had coordinates x, y and z will be:

$$\widehat{\mathbf{R}} = (x + u(x))\mathbf{i}_1 + (z)\mathbf{i}_2 + (y)\mathbf{i}_3 \tag{10}$$

The tangent and reciprocal base vectors in the deformed state will be therefore be

$$\begin{aligned} &\hat{\mathbf{g}}_1 = \frac{\partial \widehat{\mathbf{R}}}{\partial x} = (1 + u_{,x})\mathbf{i}_1 \quad \hat{\mathbf{g}}_2 = \frac{\partial \widehat{\mathbf{R}}}{\partial z} = \mathbf{i}_2 \quad \hat{\mathbf{g}}_3 = \frac{\partial \widehat{\mathbf{R}}}{\partial y} = \mathbf{i}_3 \\ &\hat{\mathbf{g}}^1 = \frac{1}{1 + u_{,x}}\mathbf{i}_1 \quad \hat{\mathbf{g}}^2 = \mathbf{i}_2 \quad \hat{\mathbf{g}}^3 = \mathbf{i}_3 \end{aligned} \tag{11}$$

where $u_{,x}$ is the derivative of $u(x)$ with respect to x. The invariants based on Eqs. (3), (4) and (11) are therefore:

$$I_\lambda = (1 + u_{,x})^2 \quad J = (1 + u_{,x}) \tag{12}$$

and the strain energy density is given by:

$$\mathrm{d}U \cong \frac{1}{2}(\Lambda + 2G)(u_{,x})^2 + \mathrm{O}([u_{,x}]^3)\,\mathrm{d}V \cong \frac{1}{2}E^{*}A(u_{,x})^2\,\mathrm{d}x \tag{13}$$

where A is the cross-sectional area. It is instructive to look at the stresses which are compatible with the assumed displacements. The first Piola–Kirchhoff stress tensor derived using Eq. (5) is (see Attard, 2003):

$$t^{ij} \cong G(g^{ir}[\delta^j_r + u^j|_r] - \hat{\mathbf{g}}^i \cdot \mathbf{g}^j) + \Lambda\hat{\mathbf{g}}^i \cdot \mathbf{g}^j(J-1) + \mathrm{O}(J-1)^2 \tag{14}$$

where δ^j_r is the kronecker delta and $u^j|_r$ is the covariant derivative of u^j. The physical Lagrangian stresses to the order of $\mathrm{O}([u_{,x}]^2)$ are therefore:

$$t^{11} \cong (2G + \Lambda)u_{,x} \quad t^{22} \cong \Lambda u_{,x} \quad t^{33} \cong \Lambda u_{,x} \quad t^{12} = t^{13} = t^{23} = 0 \tag{15}$$

The lateral stresses t^{22} and t^{33} are not insignificant as would be the case for an approximately uniaxial stress state. With hindsight a better solution can be obtained by allowing the cross-section to dilate. Let the new position vector $\widehat{\mathbf{R}}$ be:

$$\widehat{\mathbf{R}} = (x + u(x))\mathbf{i}_1 + z(1 - \mu u_{,x})\mathbf{i}_2 + y(1 - \mu u_{,x})\mathbf{i}_3 \tag{16}$$

The tangent and reciprocal base vectors in the deformed state will now be (note, it is assumed that $u_{,xx} = 0$):

$$\begin{gathered}\hat{\mathbf{g}}_1 = (1 + u_{,x})\mathbf{i}_1 \quad \hat{\mathbf{g}}_2 = (1 - \mu u_{,x})\mathbf{i}_2 \quad \hat{\mathbf{g}}_3 = (1 - \mu u_{,x})\mathbf{i}_3 \\ \hat{\mathbf{g}}^1 = \frac{1}{1 + u_{,x}}\mathbf{i}_1 \quad \hat{\mathbf{g}}^2 = \frac{1}{1 - \mu u_{,x}}\mathbf{i}_2 \quad \hat{\mathbf{g}}^3 = \frac{1}{1 - \mu u_{,x}}\mathbf{i}_3\end{gathered} \tag{17}$$

The new invariants are therefore:

$$I_\lambda = (1 + u_{,x})^2 + 2(1 - \mu u_{,x})^2 \quad J = (1 + u_{,x})(1 - \mu u_{,x})^2 \tag{18}$$

The strain energy density is then given by:

$$\mathrm{d}U \cong \frac{1}{2}E(u_{,x})^2 + \mathrm{O}([u_{,x}]^3)\,\mathrm{d}V \cong \frac{1}{2}EA(u_{,x})^2\,\mathrm{d}x \tag{19}$$

The conventional elasticity expression for uniaxial tension/compression results. The Lagrangian stresses to the order of $\mathrm{O}([u_{,x}]^2)$ are therefore:

$$t^{11} \cong Eu_{,x} \quad t^{22} \cong 0 \quad t^{33} \cong 0 \quad t^{12} = t^{13} = t^{23} = 0 \tag{20}$$

Hence, Eq. (16) provides a better approximation for the displacements under uniaxial tension and compression. The displacements described in Eq. (10) still provide an adequate solution for the longitudinal stresses if we are prepared to assume that stresses are uniaxial and replace $\Lambda + 2G$ by E in Eq. (7). It should be noted, however, that stability analysis of elastic members requires solutions to higher order. One therefore needs to be consistent in the level of approximation made.

4. Bending of a straight prismatic beam including shear and axial deformations—Timoshenko beam

This example is of the analysis of a beam under bending and axial deformation. The simplest approach is to treat the beam as a two-dimensional problem. Under deformation it is assumed that the cross-sectional shape remains unchanged (undergoes no strain within the cross-section plane) and therefore $\lambda_2 = 1$, $\lambda_3 = 1$ and $\hat{g}_{23} = 0$. As in the example of uniaxial deformation, equilibrium will therefore only be approximately satisfied as the maintenance of the cross-sectional shape implies the presences of lateral stresses to contra the effects of volume dilation.

The longitudinal axis of centroids of the undeformed beam is taken as the x- or 1-axis. The axis perpendicular to the x-axis is taken as the y- or 2-axis. The deflected shape of the beam will be characterized by the deflection of the centroidial axis. The initial axis system chosen is a Cartesian rectangular system. The material lines within the beam are assumed to be parallel to the Cartesian coordinate system. Hence the tangent base vectors at any point within the undeformed beam are $\mathbf{g}_1 = \mathbf{i}_1$ and $\mathbf{g}_2 = \mathbf{i}_2$.

Let the displacements in the x and y directions of any point within the beam be denoted by u_1 and u_2, respectively. The position vector $\widehat{\mathbf{R}}$ for a particle that initially had coordinates x, y will be:

$$\widehat{\mathbf{R}} = (x + u_1)\mathbf{i}_1 + (y + u_2)\mathbf{i}_2 \tag{21}$$

The tangent base vectors in the deformed state will be therefore be

$$\hat{\mathbf{g}}_1 = \frac{\partial \widehat{\mathbf{R}}}{\partial x} = (1 + u_{1,1})\mathbf{i}_1 + u_{2,1}\mathbf{i}_2 \quad \hat{\mathbf{g}}_2 = \frac{\partial \widehat{\mathbf{R}}}{\partial y} = u_{1,2}\mathbf{i}_1 + (1 + u_{2,2})\mathbf{i}_2 \tag{22}$$

The angle at the centroid between the tangent base vector $\hat{\mathbf{g}}_1$ in the deformed state and the undeformed longitudinal axis, is denoted by β. This angle is split into a bending component θ and a shear component φ so that $\beta = \theta + \varphi$. It is assumed that the plane of the cross-section does not remain perpendicular to the centroidal axis during deformation. In the deformed state, the angle between the plane of the cross-section and the centroidal longitudinal axis is given by θ. From Eq. (22) we can write at the centroidal axis ($y = 0$):

$$\hat{\mathbf{g}}_1 \cdot \mathbf{g}_1 = \lambda_{10} \cos\beta_0 = 1 + u_{1,1} \tag{23}$$

$$\hat{\mathbf{g}}_1 \cdot \mathbf{g}_2 = \lambda_{10} \cos\left(\frac{\pi}{2} - \beta_0\right) = u_{2,1} \tag{24}$$

$$\hat{\mathbf{g}}_2 \cdot \mathbf{g}_1 = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = u_{1,2} \tag{25}$$

$$\hat{\mathbf{g}}_2 \cdot \mathbf{g}_2 = \cos\theta = 1 + u_{2,2} \tag{26}$$

where λ_{10} is the stretch of the centroidal longitudinal axis and β_0 is defined at the centroid. Eqs. (23)–(26) lead to the following equations relating the bending angle to the displacements at the centroidal axis ($y = 0$):

$$\lambda_{10} \sin\beta_0 = u_{2,1} \quad \lambda_{10} \cos\beta_0 = 1 + u_{1,1} \quad \tan\beta_0 = \frac{u_{2,1}}{1 + u_{1,1}} \tag{27}$$

Integrating Eqs. (25) and (26) leads to expressions for the displacement functions, that is:

$$u_1 = u_o(x) - y\sin\theta \tag{28}$$

$$u_2 = v(x) - y(1 - \cos\theta) \tag{29}$$

The displacement functions are the same as for pure bending. Substituting Eqs. (28) and (29) into Eqs. (27) leads to:

$$\lambda_{10} \cos\beta_0 = 1 + u_{0,x} \quad \lambda_{10} \sin\beta_0 = v_{,x} \tag{30}$$

and

$$\tan\beta_0 = \frac{v_{,x}}{1 + u_{0,x}} \quad \frac{\mathrm{d}\beta_0}{\mathrm{d}x} = \frac{\mathrm{d}\theta}{\mathrm{d}x} + \frac{\mathrm{d}\varphi_0}{\mathrm{d}x} = \frac{(v_{,xx}[1 + u_{0,x}] - v_{,x}u_{0,xx})}{(1 + u_{0,x})^2 + v_{,x}^2} \tag{31}$$

Substituting Eqs. (28) and (29) into the equations for the tangent and reciprocal base vectors in the deformed state and using Eq. (30) we get:

$$\hat{\mathbf{g}}_1 = \lambda_1 \cos\beta \mathbf{i}_1 + \lambda_1 \sin\beta \mathbf{i}_2 = (1 + u_{0,x} - y\theta_{,x} \cos\theta)\mathbf{i}_1 + (v_{,x} - y\theta_{,x} \sin\theta)\mathbf{i}_2$$
$$= [\lambda_{10} \cos\beta_0 - y\theta_{,x} \cos\theta]\mathbf{i}_1 + [\lambda_{10} \sin\beta_0 - y\theta_{,x} \sin\theta]\mathbf{i}_2 \quad (32)$$

$$\hat{\mathbf{g}}_2 = -\sin\theta \mathbf{i}_1 + \cos\theta \mathbf{i}_2 \quad (33)$$

$$J\hat{\mathbf{g}}^1 = \cos\theta \mathbf{i}_1 + \sin\theta \mathbf{i}_2 \quad (34)$$

$$J\hat{\mathbf{g}}^2 = -\lambda_1 \sin\beta \mathbf{i}_1 + \lambda_1 \cos\beta \mathbf{i}_2 = -(v_{,x} - y\theta_{,x} \sin\theta)\mathbf{i}_1 + (1 + u_{0,x} - y\theta_{,x} \cos\theta)\mathbf{i}_2$$
$$= -[\lambda_{10} \sin\beta_0 - y\theta_{,x} \sin\theta]\mathbf{i}_1 + [\lambda_{10} \cos\beta_0 - y\theta_{,x} \cos\theta]\mathbf{i}_2 \quad (35)$$

The longitudinal tangent base vector $\hat{\mathbf{g}}_1$ is a function of the distance from the centroidal axis while the tangent base vector $\hat{\mathbf{g}}_2$ remains a function of the original x coordinate only. As in the case of pure bending, the material line that was originally perpendicular to the straight longitudinal centroidal axis remains a straight line. The angle of inclination is the bending angle θ.

The shear deformation is characterized by the scalar product of the tangent base vectors $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_2$ in the deformed state, hence using Eqs. (32) and (33):

$$\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_2 = \lambda_1 \sin\varphi = -(1 + u_{0,x})\sin\theta + v_{,x} \cos\theta = \lambda_{10} \sin\varphi_0 \quad (36)$$

The scalar product between the tangent base vectors is constant through the depth of the cross-section. The shear angle is however, not constant through the depth. The stretch λ_1 can be derived from Eq. (32) and is:

$$\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_1 = (\lambda_1)^2 = (\lambda_{10} \cos\varphi_0 - y\theta_{,x})^2 + (\lambda_{10} \sin\varphi_0)^2 = (\lambda_{10})^2 - 2\lambda_{10} \cos\varphi_0 y\theta_{,x} + y^2(\theta_{,x})^2 \quad (37)$$

where

$$\lambda_{10} = \sqrt{(1 + u_{0,x})^2 + v_{,x}^2} \quad (38)$$

The components of the stretch normal and tangential to the plane of the cross-section can be derived from Eqs. (32) and (33), and are given by the simple expressions:

$$\lambda_{n1} = \lambda_1 \cos\varphi = \lambda_{10} \cos\varphi_0 - y\theta_{,x} \quad \lambda_{s1} = \lambda_1 \sin\varphi = \lambda_{10} \sin\varphi_0 \quad (39)$$

The invariants are therefore

$$\begin{aligned} I_\lambda &= (\lambda_1)^2 + 2 = (\lambda_{10} \cos\varphi_0 - y\theta_{,x})^2 + (\lambda_{10} \sin\varphi_0)^2 + 2 \\ J &= \lambda_1 \cos\varphi = \lambda_{10} \cos\varphi_0 - y\theta_{,x} \end{aligned} \quad (40)$$

Using Eqs. (8) and (40), the strain energy density is approximated by:

$$\mathrm{d}U \cong \frac{1}{2}E^* A(\lambda_{10} \cos\varphi_0 - 1)^2 + \frac{1}{2}E^* I \left[\frac{\mathrm{d}\theta}{\mathrm{d}x}\right]^2 + \frac{1}{2}GA(\lambda_{10} \sin\varphi_0)^2 \,\mathrm{d}x \quad (41)$$

where A is the cross-section area and I is the second moment of area. This expression is similar to that derived by Simo et al. (1984a) (Eq. (25), p. 310).

The internal beam action, N the axial force normal to the cross-section, Q the shear force tangential to the cross-section and M the bending moment are obtained from the above as:

$$\begin{aligned}
&\frac{\partial \mathrm{d}U}{\partial \frac{\mathrm{d}\theta}{\mathrm{d}x}} \cong E^{*}I\frac{\mathrm{d}\theta}{\mathrm{d}x} = M \\
&\frac{\partial \mathrm{d}U}{\partial \lambda_{10}\cos\varphi_0} \cong E^{*}A(\lambda_{10}\cos\varphi_0 - 1) = N \\
&\frac{\partial \mathrm{d}U}{\partial \lambda_{10}\sin\varphi_0} = GA\lambda_{10}\sin\varphi_0 = Q
\end{aligned} \tag{42}$$

Eq. (39) is substituted into the constitutive relationships for the Reissner–Lagrangian stresses given in Eq. (9). Thus

$$\begin{aligned}
&s_{\mathrm{R}}^{\mathrm{normal}} \cong E^{*}\left(\lambda_{10}\cos\varphi_0 - 1 - y\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) \\
&s_{\mathrm{R}}^{\mathrm{shear}} = G\lambda_{10}\sin\varphi_0
\end{aligned} \tag{43}$$

5. Buckling of an initially straight prismatic column with shear and axial deformations

Consider a straight prismatic simply supported column of length L, as depicted in Fig. 4. Compressive loads P are applied at each end. The Reissner stresses act normal to the cross-section and the shears tangential to the cross-section hence from equilibrium at a free body in the column such as in Fig. 5, we can conclude the following:

$$\begin{aligned}
&\iint_A s_{\mathrm{R}}^{\mathrm{normal}}\,\mathrm{d}A \cong E^{*}A(\lambda_{10}\cos\varphi_0 - 1) = N = -P\cos\theta \\
&\iint_A s_{\mathrm{R}}^{\mathrm{normal}}y\,\mathrm{d}A \cong -E^{*}I\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = M = Pv \\
&\iint_A s_{\mathrm{R}}^{\mathrm{normal}}\,\mathrm{d}A = GA\lambda_{10}\sin\varphi_0 = Q = P\sin\theta
\end{aligned} \tag{44}$$

The equations above are combined and Eq. (30) is used to derive the following differential equation.

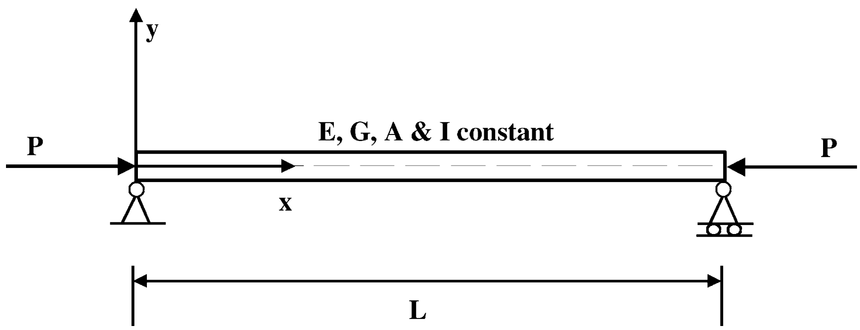


Fig. 4. Simply supported column.

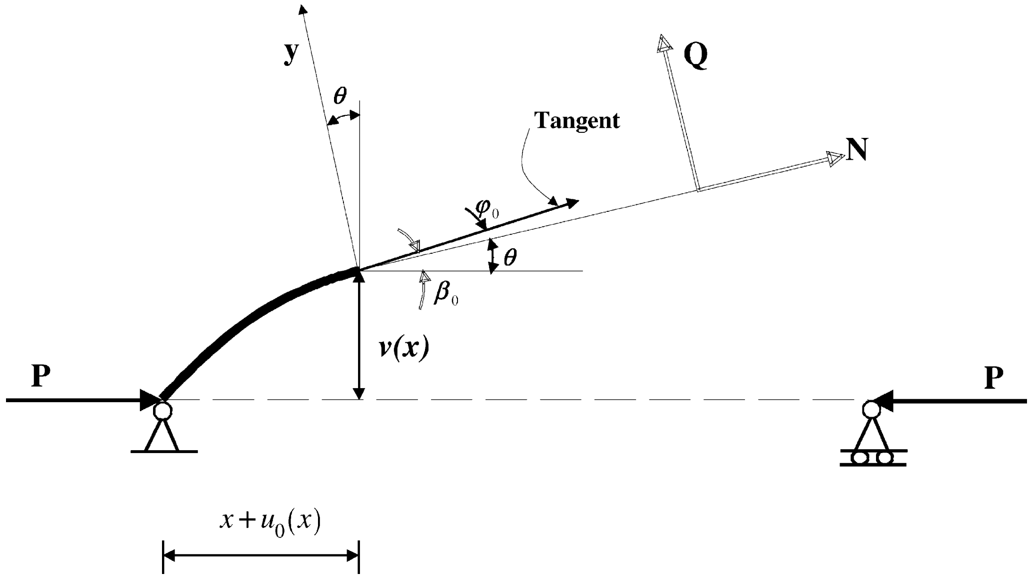


Fig. 5. Free body at the deflected centroidal axis.

$$\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right) = -\frac{P}{E^*I}v_{,x} = -\frac{P}{E^*I}\lambda_{10}\sin\beta_0 = -\lambda_{10}\sin\varphi_0\frac{P}{E^*I}\cos\theta - \lambda_{10}\cos\varphi_0\frac{P}{E^*I}\sin\theta$$
$$= -\frac{P}{E^*A}\frac{1}{r^2}\sin\theta + \left(\frac{P}{E^*A}\right)^2\frac{1}{r^2}\cos\theta\sin\theta\left(1-\frac{E^*}{G}\right) = \frac{1}{r^2}P^*\sin\theta[P^*\cos\theta(1-m^*)-1] \tag{45}$$

where r is the radius of gyration and $P^* = P^*/E^*A$ and $m^* = E^*/G$. An estimate of the buckling load can be derived from this differential equation. The details are contained in Appendix A. The resulting formula for the buckling load P_{cr} is:

$$\frac{P_{\mathrm{cr}}}{E^*A} = \frac{1}{2(m^*-1)}\left\{-1+\sqrt{1+\frac{4\pi^2}{(L/r)^2}(m^*-1)}\right\} \tag{46}$$

This equation is identical to the second equation derived by Timoshenko and Gere (1963, p. 143) for this problem. The above equation is also very similar to the equation derived by Haringx, written here as:

$$\frac{P_{\mathrm{cr}}}{EA} = \frac{1}{2m}\left\{-1+\sqrt{1+\frac{4\pi^2}{(L/r)^2}m}\right\} \tag{47}$$

where $m = E/G$.

6. Bending and torsion of a straight three-dimensional prismatic cylindrical beam including shear—Timoshenko beam

Consider a straight cylindrical prismatic beam. Under bending and torsion deformations it is assumed that the cross-sectional shape remains unchanged (undergoes no strain within the cross-section plane) and there is no cross-sectional warping. The initial axis system chosen is a Cartesian rectangular system and

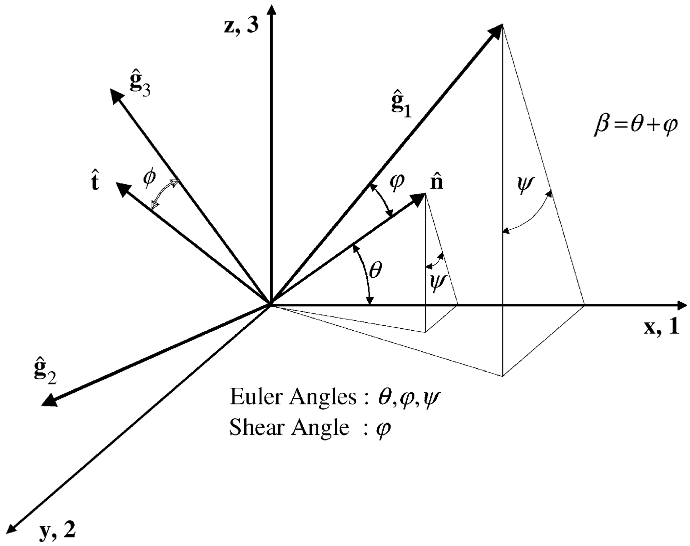


Fig. 6. Tangent base vectors in three-dimensions.

there is therefore no distinction between covariant and contravariant differentiation of tensor components. The material lines within the beam are parallel to the Cartesian coordinate system. The longitudinal axis of centroids of the undeformed beam is taken as the x- or 1-axis. The cross-sectional centroidal principal axes are taken as the y- or 2-axis and the z- or 3-axis (refer to Fig. 6). Therefore λ_2 and $\lambda_3 = 1$ and $\hat{\mathbf{g}}_2 \cdot \hat{\mathbf{g}}_3 = 0$. As with the previous two-dimensional beam problem, equilibrium will only be approximately satisfied because the cross-sectional shape is constrained.

The deflected shape of the beam will be characterized by the deflection of the centroidal axis. The tangent base vectors at any point within the undeformed beam are:

$$\mathbf{g}_1 = \mathbf{i}_1 \quad \mathbf{g}_2 = \mathbf{i}_2 \quad \mathbf{g}_3 = \mathbf{i}_3 \tag{48}$$

Let the displacements in the x, y and z directions of any point within the beam be denoted by u_1, u_2 and u_3, respectively. The position vector $\widehat{\mathbf{R}}$ for a particle that initially had coordinates x, y and z will be:

$$\widehat{\mathbf{R}} = (x + u_1)\mathbf{i}_1 + (y + u_2)\mathbf{i}_2 + (z + u_3)\mathbf{i}_3 \tag{49}$$

The tangent base vectors in the deformed state are therefore defined by

$$\begin{aligned} \hat{\mathbf{g}}_1 &= \frac{\partial \widehat{\mathbf{R}}}{\partial x} = (1 + u_{1,1})\mathbf{i}_1 + u_{2,1}\mathbf{i}_2 + u_{3,1}\mathbf{i}_3 = \lambda_1(l_1\mathbf{i}_1 + m_1\mathbf{i}_2 + n_1\mathbf{i}_3) \\ \hat{\mathbf{g}}_2 &= \frac{\partial \widehat{\mathbf{R}}}{\partial y} = u_{1,2}\mathbf{i}_1 + (1 + u_{2,2})\mathbf{i}_2 + u_{3,2}\mathbf{i}_3 = l_2\mathbf{i}_1 + m_2\mathbf{i}_2 + n_2\mathbf{i}_3 \\ \hat{\mathbf{g}}_3 &= \frac{\partial \widehat{\mathbf{R}}}{\partial z} = u_{1,3}\mathbf{i}_1 + u_{2,3}\mathbf{i}_2 + (1 + u_{3,3})\mathbf{i}_3 = l_3\mathbf{i}_1 + m_3\mathbf{i}_2 + n_3\mathbf{i}_3 \end{aligned} \tag{50}$$

where l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2 and n_3 are direction cosines. The angle at the centroid between the tangent base vector $\hat{\mathbf{g}}_1$ in the deformed state and the undeformed longitudinal axis $\mathbf{g}_1$, is denoted by β. This angle is assumed to consist of a bending component θ and a shear component φ so that $\beta = \theta + \varphi$. It is also assumed that the plane of the cross-section does not remain perpendicular to the centroidal axis during deformation. The unit normal to the deformed cross-sectional plane (containing $\hat{\mathbf{g}}_2$ and $\hat{\mathbf{g}}_3$) at the centroid is denoted by $\hat{\mathbf{n}}$ and is defined by

$$\hat{\mathbf{n}} = l_{10}\mathbf{i}_1 + m_{10}\mathbf{i}_2 + n_{10}\mathbf{i}_3 \tag{51}$$

where l_{10}, m_{10} and n_{10} are direction cosines. The unit normal $\hat{\mathbf{n}}$ lies in the plane containing $\hat{\mathbf{g}}_1$ and $\mathbf{g}_1$ as shown in Fig. 6 and satisfies the following:

$$\begin{aligned}
\hat{\mathbf{n}}_1 \cdot \mathbf{g}_1 &= l_{10} = \cos\theta \\
\hat{\mathbf{n}}_1 \cdot \mathbf{g}_2 &= m_{10} = \sin\theta\sin\psi \\
\hat{\mathbf{n}}_1 \cdot \mathbf{g}_3 &= n_{10} = \sin\theta\cos\psi
\end{aligned} \tag{52}$$

where ψ is angle between the plane containing $\hat{\mathbf{g}}_1$ and $\mathbf{g}_1$ and the z axis (refer to Fig. 6) and is related to the tortuosity of the centroidal deformed axis (see Love, 1944). Since $\hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3$ and $\hat{\mathbf{n}}$ are all orthogonal we can use a system of Euler angles θ, ψ and ϕ to define their direction cosines. Hence at the centroidal axis (y and $z = 0$), we have the following equations:

$$\begin{aligned}
\hat{\mathbf{g}}_1 \cdot \mathbf{g}_1 &= 1 + u_{1,1} = \lambda_{10} l_1 = \lambda_{10}\cos\beta_0 \\
\hat{\mathbf{g}}_1 \cdot \mathbf{g}_2 &= u_{2,1} = \lambda_{10} m_1 = \lambda_{10}\sin\beta_0\sin\psi \\
\hat{\mathbf{g}}_1 \cdot \mathbf{g}_3 &= u_{3,1} = \lambda_{10} n_1 = \lambda_{10}\sin\beta_0\cos\psi
\end{aligned} \tag{53}$$

$$\begin{aligned}
\hat{\mathbf{g}}_2 \cdot \mathbf{g}_1 &= u_{1,2} = l_2 = \sin\theta\sin\phi \\
\hat{\mathbf{g}}_2 \cdot \mathbf{g}_2 &= 1 + u_{2,2} = m_2 = \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta \\
\hat{\mathbf{g}}_2 \cdot \mathbf{g}_3 &= u_{3,2} = n_2 = -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta
\end{aligned} \tag{54}$$

$$\begin{aligned}
\hat{\mathbf{g}}_3 \cdot \mathbf{g}_1 &= u_{1,3} = l_3 = -\sin\theta\cos\phi \\
\hat{\mathbf{g}}_3 \cdot \mathbf{g}_2 &= u_{2,3} = m_3 = \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta \\
\hat{\mathbf{g}}_3 \cdot \mathbf{g}_3 &= 1 + u_{3,3} = n_3 = -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta
\end{aligned} \tag{55}$$

where the angles β_0, ψ and ϕ are taken as functions of x only. Eq. (53) leads to the following

$$\begin{aligned}
&\lambda_{10}\sin\beta_0\sin\psi = v_{,x} \quad \lambda_{10}\sin\beta_0\cos\psi = w_{,x} \\
&\lambda_{10}\cos\beta_0 = 1 + u_{0,x} \quad \lambda_{10}\sin\beta_0 = \sqrt{v_{,x}^2 + w_{,x}^2} \\
&\lambda_{10} = \sqrt{(1 + u_0,x)^2 + v_{,x}^2 + w_{,x}^2}
\end{aligned} \tag{56}$$

where $u_0(x)$ is the axial displacement of the centroidal axis and, $v(x)$ and $w(x)$ are the displacements of the centroidal axis in the y and z directions, respectively, and λ_{10} is the stretch of the centroidal axis. Eqs. (56) lead to the following equations relating the bending angle and tortuosity angle to the displacements at the centroidal axis (y and $z = 0$):

$$\begin{aligned}
&\frac{w_{,x}}{1 + u_{0,x}} = \tan\beta_0\cos\psi \quad \frac{v_{,x}}{1 + u_{0,x}} = \tan\beta_0\sin\psi \\
&\frac{v_{,x}}{w_{,x}} = \tan\psi \quad \tan\beta_0 = \frac{\sqrt{w_{,x}^2 + v_{,x}^2}}{1 + u_{0,x}}
\end{aligned} \tag{57}$$

The geometric torsion or tortuosity of the centroidal axis is related to the derivative of the angle ψ with respect to x, and can be derived from the above equations, that is:

$$\frac{d\psi}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{v_{,x}}{w_{,x}}\right) = \frac{w_{,x}v_{,xx} - v_{,x}w_{,xx}}{w_{,x}^2 + v_{,x}^2} \tag{58}$$

Integrating Eqs. (54) and (55) leads to expressions for the displacement functions, that is:

$$\begin{aligned} u_1 &= u_0(x) + yl_2 + zl_3 \\ u_2 &= v(x) - y(1 - m_2) + zm_3 \\ u_3 &= w(x) + yn_2 - z(1 - n_3) \end{aligned} \tag{59}$$

The curvatures with respect to the unit normal $\hat{\mathbf{n}}$ at the centroid are given by

$$\kappa = -\hat{\mathbf{n}}_{,x} \cdot \hat{\mathbf{g}}_2 = \frac{d\theta}{dx}\sin\phi - \frac{d\psi}{dx}\sin\theta\cos\phi \tag{60}$$

$$\kappa' = \hat{\mathbf{n}}_{,x} \cdot \hat{\mathbf{g}}_3 = \frac{d\theta}{dx}\cos\phi + \frac{d\psi}{dx}\sin\theta\sin\phi \tag{61}$$

while the torsion at the centroid about the unit normal to the cross-section is defined by

$$\tau = \hat{\mathbf{g}}_{3,x} \cdot \hat{\mathbf{g}}_2 = \frac{d\phi}{dx} + \frac{d\psi}{dx}\cos\theta \tag{62}$$

Following the procedure in Love (1944), the tangent base vector $\hat{\mathbf{g}}_1$ can now be written in terms of the curvatures and torsion of the centroidal axis:

$$\begin{aligned} \hat{\mathbf{g}}_1 = {} & (\lambda_{10}\cos\beta_0 + y[l_{10}\kappa - l_3\tau] + z[l_2\tau - l_{10}\kappa'])\mathbf{i}_1 + (\lambda_{10}\sin\beta_0\sin\psi + y[m_{10}\kappa - m_3\tau] \\ & + z[m_2\tau - m_{10}\kappa'])\mathbf{i}_2 + (\lambda_{10}\sin\beta_0\cos\psi + y[n_{10}\kappa - n_3\tau] + z[n_2\tau - n_{10}\kappa'])\mathbf{i}_3 \end{aligned} \tag{63}$$

The shear deformation is characterized by the scalar product of the tangent base $\hat{\mathbf{g}}_1$ with $\hat{\mathbf{g}}_2$ and $\hat{\mathbf{g}}_3$ separately, in the deformed state, hence using Eqs. (50), (54), (55) and (63) we have:

$$\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_2 = z\tau - \lambda_{10}\sin\phi\sin\varphi_0 \quad \hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_3 = -y\tau + \lambda_{10}\cos\phi\sin\varphi_0 \tag{64}$$

The stretch λ_1 can be derived from Eq. (63) and is:

$$\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_1 = (\lambda_1)^2 = (\lambda_{10}\cos\varphi_0 + y\kappa - z\kappa')^2 + (z\tau - \lambda_{10}\sin\phi\sin\varphi_0)^2 + (-y\tau + \lambda_{10}\cos\phi\sin\varphi_0)^2 \tag{65}$$

consisting of the square of the normal component of the longitudinal stretch and the two components associated with shear. The invariants are therefore:

$$\begin{aligned} I_\lambda &= (\lambda_1)^2 + 2 = (\lambda_{10}\cos\varphi_0 + y\kappa - z\kappa')^2 \\ &\quad + (z\tau - \lambda_{10}\sin\phi\sin\varphi_0)^2 + (-y\tau + \lambda_{10}\cos\phi\sin\varphi_0)^2 + 2 \\ J &= \lambda_1(1 - \cos^2\hat{\phi}_{12} - \cos^2\hat{\phi}_{13})^{1/2} = \lambda_{10}\cos\varphi + y\kappa - z\kappa' \end{aligned} \tag{66}$$

For small strain, the strain energy density becomes

$$\begin{aligned} dU \cong {} & \frac{1}{2}E^*A(\lambda_{10}\cos\varphi_0 - 1)^2 + \frac{1}{2}GA\left([\lambda_{10}\sin\varphi_0]^2 + \frac{I_{po}}{A}\tau^2\right) + \frac{1}{2}E^*(I_{zz}\kappa^2 + I_{yy}[\kappa']^2)\,dx \\ & + \int_A O([\lambda_{10}\cos\varphi_0 + y\kappa - z\kappa' - 1]^3)\,dA\,dx \end{aligned} \tag{67}$$

where I_{zz} and I_{yy} are the second moment of areas about the y and z axes, respectively and I_{po} is the polar second moment of area. The internal beam actions obtained from Eq. (67) are therefore:

$$
\begin{aligned}
&\frac{\partial \mathrm{d}U}{\partial \kappa} \cong E^* I_{zz}\kappa = M_3 \\
&\frac{\partial \mathrm{d}U}{\partial \kappa'} \cong E^* I_{yy}\kappa' = M_2 \\
&\frac{\partial \mathrm{d}U}{\partial \tau} = G I_{\mathrm{po}}\tau = M_{\mathrm{t}} \\
&\frac{\partial \mathrm{d}U}{\partial \lambda_{10}\cos\varphi_0} \cong E^* A(\lambda_{10}\cos\varphi_0 - 1) = N \\
&\frac{\partial \mathrm{d}U}{\partial \lambda_{10}\sin\varphi_0} = GA\lambda_{10}\sin\varphi_0 = Q
\end{aligned} \tag{68}
$$

In the above equations, M_3 and M_2 are the resultant bending moments about the deformed $\hat{\mathbf{g}}_3$ and $\hat{\mathbf{g}}_2$ axes, respectively; M_t is the twisting moment about the unit normal to the cross-section; N is the axial force normal to the deformed cross-sectional plane in the direction $\hat{\mathbf{n}}$ and, Q is the shear force resultant which acts in the plane of $\hat{\mathbf{g}}_1$ and $\mathbf{g}_1$ and perpendicular to $\hat{\mathbf{n}}$ (refer to Fig. 6), defined by the unit vector $\hat{\mathbf{t}}$ given by

$$\hat{\mathbf{t}} = -\sin\theta \mathbf{i}_1 + \cos\theta\sin\psi \mathbf{i}_2 + \cos\theta\cos\psi \mathbf{i}_3 \tag{69}$$

The Reissner–Lagrangian stress representation is convenient for describing the stresses on the cross-section and are:

$$
\begin{aligned}
&s_{\mathrm{R}}^{\mathrm{normal}} \cong E^*(\lambda_{10}\cos\varphi_0 + y\kappa - z\kappa' - 1) \\
&s_{\mathrm{R}}^{12} = G(z\tau - \lambda_{10}\sin\phi\sin\varphi_0) \\
&s_{\mathrm{R}}^{13} = G(-y\tau + \lambda_{10}\cos\phi\sin\varphi_0)
\end{aligned} \tag{70}
$$

7. Uniform torsion of a cylinder and the Wagner effect

The next example is that of a cylinder under uniform torsion and is examined because the results differ from conventional theory which shows a second-order axial shortening of the cylinder under pure torsion (see Timoshenko, 1953, p. 402). This second-order axial shortening is important in the derivation of torsional and flexural–torsional buckling and is associated with the Wagner effect (see Attard, 1986; Alwis and Wang, 1996). Although polar coordinates can be used, Cartesian coordinates will be used to simplify the analysis. The longitudinal axis of the cylinder is denoted by x while the other two principal axes are denoted by y and z. Consider a point within the cylinder which initially had coordinates y and z. The cylinder is twisted through an angle ϕ about the longitudinal axis and with hindsight the vector representing the deformed point is chosen as:

$$\widehat{\mathbf{R}} = [x + u(x)]\mathbf{i}_1 + \bar{z}(1 - \mu u_{,x})\mathbf{i}_2 + \bar{y}(1 - \mu u_{,x})\mathbf{i}_3 \tag{71}$$

where $\bar{z}$ and $\bar{y}$ are the rotated coordinates defined by:

$$\bar{z} = (z\cos\phi - y\sin\phi) \quad \bar{y} = (z\sin\phi + y\cos\phi) \tag{72}$$

Points within any cross-section will rotate about the longitudinal axis and suffer a longitudinal displacement only if $u(x)$ is non-zero. Only uniform axial displacement and uniform torsion is considered here ($u_{,xx} = 0$ and $\phi_{,xx} = 0$). The tangent and reciprocal base vectors in the deformed state will therefore be

$$\begin{aligned}\hat{\mathbf{g}}_1 &= \frac{\partial\widehat{\mathbf{R}}}{\partial x} = (1+u_{,x})\mathbf{i}_1 - \bar{y}(1-\mu u_{,x})\phi_{,x}\mathbf{i}_2 + \bar{z}(1-\mu u_{,x})\phi_{,x}\mathbf{i}_3\\ \hat{\mathbf{g}}_2 &= \frac{\partial\widehat{\mathbf{R}}}{\partial z} = (\cos\phi\mathbf{i}_2 + \sin\phi\mathbf{i}_3)(1-\mu u_{,x})\\ \hat{\mathbf{g}}_3 &= \frac{\partial\widehat{\mathbf{R}}}{\partial y} = (-\sin\phi\mathbf{i}_2 + \cos\phi\mathbf{i}_3)(1-\mu u_{,x})\end{aligned} \tag{73}$$

$$\begin{aligned}\hat{\mathbf{g}}^1 &= \frac{1}{(1+u_{,x})}\mathbf{i}_1\\ \hat{\mathbf{g}}^2 &= \frac{y\phi_{,x}}{(1+u_{,x})}\mathbf{i}_1 + \frac{(\cos\phi\mathbf{i}_2+\sin\phi\mathbf{i}_3)}{(1-\mu u_{,x})}\\ \hat{\mathbf{g}}^3 &= \frac{-z\phi_{,x}}{(1+u_{,x})}\mathbf{i}_1 + \frac{(-\sin\phi\mathbf{i}_2+\cos\phi\mathbf{i}_3)}{(1-\mu u_{,x})}\end{aligned} \tag{74}$$

giving rise to the following results:

$$\begin{aligned}&\hat{\mathbf{g}}_1\cdot\hat{\mathbf{g}}_1 = (\lambda_1)^2 = (1+u_{,x})^2 + r^2(\phi_{,x})^2(1-\mu u_{,x})^2\\ &\hat{\mathbf{g}}_2\cdot\hat{\mathbf{g}}_2 = \hat{\mathbf{g}}_3\cdot\hat{\mathbf{g}}_3 = (1-\mu u_{,x})^2\\ &\hat{\mathbf{g}}_2\cdot\hat{\mathbf{g}}_3 = 0 \quad \therefore\ \hat{\phi}_{23} = \frac{\pi}{2}\\ &\hat{\mathbf{g}}_1\cdot\hat{\mathbf{g}}_2 = \lambda_1\cos\hat{\phi}_{12} = -y\phi_{,x}(1-\mu u_{,x})^2\\ &\hat{\mathbf{g}}_1\cdot\hat{\mathbf{g}}_3 = \lambda_1\cos\hat{\phi}_{13} = z\phi_{,x}(1-\mu u_{,x})^2\end{aligned} \tag{75}$$

where $r^2 = z^2 + y^2$. The invariants are therefore:

$$\begin{aligned}I_\lambda &= (1+u_{,x})^2 + r^2(\phi_{,x})^2(1-\mu u_{,x})^2 + 2(1-\mu u_{,x})^2\\ J &= \lambda_1(1-\cos^2\hat{\phi}_{12}-\cos^2\hat{\phi}_{13})^{1/2} = (1+u_{,x})(1-\mu u_{,x})^2\end{aligned} \tag{76}$$

The normal and shear components of the longitudinal stretch are therefore:

$$\lambda_{n1} = 1 + u_{,x} \quad \lambda_{s1} = r\phi_{,x}(1-\mu u_{,x}) \tag{77}$$

The normal component of the longitudinal stretch is unaffected by the twist as is also the invariant J. The strain energy density based on Eq. (7) is then:

$$\mathrm{d}U \cong \tfrac{1}{2}GI_{\mathrm{po}}(1-\mu u_{,x})^2(\phi_{,x})^2 + \tfrac{1}{2}EA(u_{,x})^2 + \mathrm{O}([u_{,x}]^3)\,\mathrm{d}x \tag{78}$$

where I_{po} is the polar second moment of area. Based on Eq. (14), the shear stress components t^{12} and t^{13} acting in the original z and y directions are:

$$t^{12} = -G\bar{y}\phi_{,x}(1-\mu u_{,x}) \quad t^{13} = G\bar{z}\phi_{,x}(1-\mu u_{,x}) \tag{79}$$

Hence, the twisting moment M_{t1} due to St. Venant shear stresses, is then

$$M_{\mathrm{t1}} \cong \int_A (t^{13}\bar{z} - t^{12}\bar{y})(1-\mu u_{,x})\,\mathrm{d}A \cong GI_{\mathrm{po}}\phi_{,x}(1-\mu u_{,x})^2 \tag{80}$$

This agrees with Eq. (78) where the same result can be obtained by noting that the twisting moment is conjugate to the rate of twist $\phi_{,x}$. The stresses normal to the cross-sectional plane of the bar are:

$$t^{11} \cong G(1 + u_{1,1} - \hat{\mathbf{g}}^1 \cdot \mathbf{g}^1) + \Lambda \hat{\mathbf{g}}^1 \cdot \mathbf{g}^1 (J-1) + \mathrm{O}([J-1]^2)$$
$$\cong G\left(1 + u_{,x} - \frac{1}{1+u_{,x}}\right) + \Lambda \frac{1}{1+u_{,x}}[(1+u_{,x})(1-\mu u_{,x})^2 - 1] \cong E u_{,x} + \mathrm{O}(u_{,x})^2 \tag{81}$$

The stresses t^{22} and t^{33} can be shown to be of second order. The axial force N is then calculated from

$$N = \int_A t^{11}\,\mathrm{d}A \cong EAu_{,x} + \mathrm{O}([u_{,x}]^2) \tag{82}$$

If there is no axial force, there is no axial shortening. Many finite strain formulations predict an axial shortening accompanied with a self-equilibrating normal stress distribution under pure torsion. This is not predicted here as the component of the stretch λ_1 normal to the cross-sectional plane is unity under pure torsion (refer to Eq. (77)) and therefore results in no normal stress. The deformation involved in pure torsion of a cylinder is essentially that of simple shear which as shown in Attard (2003) produces no normal stress.

8. Uniform torsion with warping and the Wagner effect

Here we extend the previous example by considering a prismatic bar of general but symmetric cross-section which under uniform torsion ($\phi_{,xx} = 0$) experiences longitudinal warping. It is assumed that the longitudinal displacement of the previous example is augmented by a warping of the cross-sectional plane which is proportional to the product of a warping function $\omega(y,z)$ and the rate of change of the twist angle $\phi_{,x}$. The cross-section is twisted about the centroidal longitudinal axis. The vector representing the position of a point (x,y,z) in the deformed state is then given by:

$$\widehat{\mathbf{R}} = [x + u(x) + \omega\phi_{,x}]\mathbf{i}_1 + \bar{z}(1-\mu u_{,x})\mathbf{i}_2 + \bar{y}(1-\mu u_{,x})\mathbf{i}_3 \tag{83}$$

with the displacements defined by

$$u_1 = u(x) + \omega\phi_{,x} \quad u_2 = \bar{z}(1-\mu u_{,x}) - z \quad u_3 = \bar{y}(1-\mu u_{,x}) - y \tag{84}$$

The associated covariant tangent base vectors in the deformed state are:

$$\hat{\mathbf{g}}_1 = \frac{\partial\widehat{\mathbf{R}}}{\partial x} = (1+u_{,x})\mathbf{i}_1 - \bar{y}\phi_{,x}(1-\mu u_{,x})\mathbf{i}_2 + \bar{z}\phi_{,x}(1-\mu u_{,x})\mathbf{i}_3$$
$$\hat{\mathbf{g}}_2 = \frac{\partial\widehat{\mathbf{R}}}{\partial z} = \omega_{,z}\phi_{,x}\mathbf{i}_1 + (\cos\phi\,\mathbf{i}_2 + \sin\phi\,\mathbf{i}_3)(1-\mu u_{,x}) \tag{85}$$
$$\hat{\mathbf{g}}_3 = \frac{\partial\widehat{\mathbf{R}}}{\partial y} = \omega_{,y}\phi_{,x}\mathbf{i}_1 + (-\sin\phi\,\mathbf{i}_2 + \cos\phi\,\mathbf{i}_3)(1-\mu u_{,x})$$

and the contravariant reciprocal base vectors given by:

$$J\hat{\mathbf{g}}^1 = [(1-\mu u_{,x})\mathbf{i}_1 + (\omega_{,y}\sin\phi - \omega_{,z}\cos\phi)\phi_{,x}\mathbf{i}_2 - (\omega_{,y}\cos\phi + \omega_{,z}\sin\phi)\phi_{,x}\mathbf{i}_3](1-\mu u_{,x})$$
$$J\hat{\mathbf{g}}^2 = [y\phi_{,x}(1-\mu u_{,x})\mathbf{i}_1 + (\cos\phi[1+u_{,x}] - \bar{z}\omega_{,y}\phi_{,x}^2)\mathbf{i}_2 + (\sin\phi[1+u_{,x}] - \bar{y}\omega_{,y}\phi_{,x}^2)\mathbf{i}_3](1-\mu u_{,x}) \tag{86}$$
$$J\hat{\mathbf{g}}^3 = [-z\phi_{,x}(1-\mu u_{,x})\mathbf{i}_1 + (-\sin\phi[1+u_{,x}] + \bar{z}\omega_{,z}\phi_{,x}^2)\mathbf{i}_2 + (\cos\phi[1+u_{,x}] + \bar{y}\omega_{,z}\phi_{,x}^2)\mathbf{i}_3](1-\mu u_{,x})$$

The dot products of the covariant tangent base vectors are therefore:

$$\begin{aligned}
\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_1 &= (\lambda_1)^2 = (1+u_{,x})^2 + r^2(\phi_{,x})^2(1-\mu u_{,x})^2 \\
\hat{\mathbf{g}}_2 \cdot \hat{\mathbf{g}}_2 &= (\lambda_2)^2 = (1-\mu u_{,x})^2 + (\omega_{,z}\phi_{,x})^2 \\
\hat{\mathbf{g}}_3 \cdot \hat{\mathbf{g}}_3 &= (\lambda_3)^2 = (1-\mu u_{,x})^2 + (\omega_{,y}\phi_{,x})^2 \\
\hat{\mathbf{g}}_2 \cdot \hat{\mathbf{g}}_3 &= \lambda_2\lambda_3 \cos\hat{\phi}_{23} = \omega_{,z}\omega_{,y}(\phi_{,x})^2 \\
\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_2 &= \lambda_1\lambda_2 \cos\hat{\phi}_{12} = (1+u_{,x})\omega_{,z}\phi_{,x} - y\phi_{,x}(1-\mu u_{,x})^2 \\
\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{g}}_3 &= \lambda_1\lambda_3 \cos\hat{\phi}_{13} = (1+u_{,x})\omega_{,y}\phi_{,x} + z\phi_{,x}(1-\mu u_{,x})^2
\end{aligned} \tag{87}$$

The first and third invariants can now be derived and are given by:

$$I_\lambda = (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 = (1+u_{,x})^2 + 2(1-\mu u_{,x})^2 + (r^2(1-\mu u_{,x})^2 + \omega_{,z}^2 + \omega_{,y}^2)\phi_{,x}^2 \tag{88}$$

$$\begin{aligned}
J &= \lambda_1\lambda_2\lambda_3(1 + 2\cos\hat{\phi}_{12}\cos\hat{\phi}_{13}\cos\hat{\phi}_{23} - \cos^2\hat{\phi}_{12} - \cos^2\hat{\phi}_{13} - \cos^2\hat{\phi}_{23})^{1/2} \\
&= (1 + u_{,x} + \bar{\omega}\phi_{,x}^2)(1-\mu u_{,x})^2
\end{aligned} \tag{89}$$

where $\bar{\omega} = y\omega_{,z} - z\omega_{,y}$. Because the plane of the cross-section warps, it is convenient to use the first Piola–Kirchhoff stress tensor to describe stresses. Since the initial coordinates are Cartesian, the first Piola–Kirchhoff stress tensor will be equal to it's physical counterpart. Using Eq. (14), the shear stress components t^{12} and t^{13} acting in the original z and y directions are:

$$\begin{aligned}
t^{12} &\cong G(u_{2,1} - \hat{\mathbf{g}}^1 \cdot \mathbf{g}^2) + \Lambda\hat{\mathbf{g}}^1 \cdot \mathbf{g}^2(J-1) + \mathrm{O}([J-1]^2) \\
&= G\phi_{,x}(1-\mu u_{,x})\left(-\bar{y} - \frac{1}{J}[-\omega_{,z}\cos\phi + \omega_{,y}\sin\phi]\right) \\
&\quad + \Lambda\phi_{,x}(1-\mu u_{,x})(-\omega_{,z}\cos\phi + \omega_{,y}\sin\phi)\left(1-\frac{1}{J}\right) \\
t^{13} &\cong G(u_{3,1} - \hat{\mathbf{g}}^1 \cdot \mathbf{g}^3) + \Lambda\hat{\mathbf{g}}^1 \cdot \mathbf{g}^3(J-1) + \mathrm{O}([J-1]^2) \\
&= G\phi_{,x}(1-\mu u_{,x})\left(\bar{z} + \frac{1}{J}[\omega_{,z}\sin\phi + \omega_{,y}\cos\phi]\right) \\
&\quad - \Lambda\phi_{,x}(1-\mu u_{,x})(\omega_{,z}\sin\phi + \omega_{,y}\cos\phi)\left(1-\frac{1}{J}\right)
\end{aligned} \tag{90}$$

These stresses do not include any shear stresses resulting from restrained warping as the displacement functions defined in Eq. (84) do not include this effect. Because there must be no resultant shear forces on the cross-section then $\int_A t^{12}\,\mathrm{d}A = 0$ and $\int_A t^{13}\,\mathrm{d}A = 0$ and several equations involving the warping function must be satisfied, some of which are

$$\begin{aligned}
&\int_A \omega_{,z}\,\mathrm{d}A = 0 \quad \int_A \omega_{,y}\,\mathrm{d}A = 0 \quad \int_A \omega\omega_{,z}\,\mathrm{d}A = 0 \quad \int_A \omega\omega_{,y}\,\mathrm{d}A = 0 \\
&\int_A \bar{\omega}\omega_{,z}\,\mathrm{d}A = 0 \quad \int_A \bar{\omega}\omega_{,y}\,\mathrm{d}A = 0 \quad \int_A \bar{\omega}z\,\mathrm{d}A = 0 \quad \int_A \bar{\omega}y\,\mathrm{d}A = 0
\end{aligned} \tag{91}$$

The twisting moment M_{t1} due to St. Venant shear stresses, can now be derived and is

$$M_{t1} \cong \int_A (t^{13}\bar{z} - t^{12}\bar{y})(1-\mu u_{,x})\,\mathrm{d}A \cong [GJ_t + E^{**}I_{\bar{\omega}}u_{,x}]\phi_{,x}(1-\mu u_{,x})^2 \tag{92}$$

where $E^{**} = G + \Lambda$ and

$$I_{\bar{\omega}} = \int_A \bar{\omega}\,\mathrm{d}A \quad J_\mathrm{t} = \int_A (r^2 - \bar{\omega})\,\mathrm{d}A = I_\mathrm{po} - I_{\bar{\omega}} \tag{93}$$

and J_t is the St. Venant torsion constant. A different formula for the torsion constant is derived if one inspects the strain energy expression and is:

$$J_\mathrm{t} = \int_A [y - \omega_{,z}]^2 + [z + \omega_{,y}]^2\,\mathrm{d}A \tag{94}$$

Eqs. (93) and (94) imply:

$$I_{\bar{\omega}} = \int_A \omega_{,z}^2 + \omega_{,y}^2\,\mathrm{d}A \tag{95}$$

The twisting moment expression has a coupling term of second-order involving the axial displacement. This implies that the torsional stiffness is affected by the presence of axial deformation. The stresses aligned with the undeformed longitudinal axis of the bar are:

$$\begin{aligned} t^{11} &\cong G(1 + u_{1,1} - \hat{\mathbf{g}}^1 \cdot \mathbf{g}^1) + \Lambda \hat{\mathbf{g}}^1 \cdot \mathbf{g}^1 (J-1) + \mathrm{O}([J-1]^2) \\ &= G\left(1 + u_{,x} - \frac{(1 - \mu u_{,x})^2}{J}\right) + \Lambda \frac{(1 - \mu u_{,x})^2}{J}(J-1) \end{aligned} \tag{96}$$

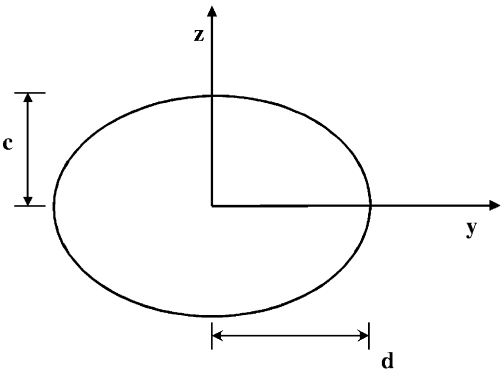
The axial force expression is then to first-order terms in $u_{,x}$ and second-order terms in $\phi_{,x}$:

$$N = \int_A t^{11}\,\mathrm{d}A \cong EAu_{,x} + E^{**} I_{\bar{\omega}} \phi_{,x}^2 \tag{97}$$

With the presence of warping has come a coupling between the axial displacement of the centroidal axis and the twist rate. If there is no axial force then there could be an axial displacement under torsion if the section warps. When there is uniform torsion without any axial force Eq. (97) gives for the axial shortening:

$$u_{,x} \cong -\frac{1}{2}\frac{1}{(1+\mu)(1-2\mu)}\frac{I_{\bar{\omega}}}{A}\phi_{,x}^2 = -\frac{1}{2}\frac{1}{(1+\mu)(1-2\mu)}\frac{I_\mathrm{po} - J_\mathrm{t}}{A}\phi_{,x}^2 \tag{98}$$

Conventional beam theory gives for the second-order axial shortening associated with the Wagner effect as $u_{,x} = -(1/2)(I_\mathrm{po}/A)\phi_{,x}^2$ (see Attard, 1986; Alwis and Wang, 1996), which is independent of the amount of warping. For comparison, consider an elliptic cross-section as shown in Fig. 7. The torsion constant for an elliptic cross-section is derived in Timoshenko and Goodier (1970) and is given in Fig. 7. Fig. 8 shows a comparison of the axial shortening calculated from Eq. (98) for various Poisson's ratio and that using the conventional Wagner expression as a function of the dimension ratio c/d. The shortening factor is the ratio of Eq. (98) to the axial shortening calculated using $u_{,x} = -(1/2)(I_\mathrm{po}/A)\phi_{,x}^2$. It is seen that the axial shortening predicted by Eq. (98) is within one or two times that predicted by the Wagner expression for large c/d ratios and for materials with Poisson's ratio less than 0.3. Since the axial shortening under pure torsion is of second-order smallness it would be difficult to discern which theory is correct except perhaps if one was able to test a cylinder under pure torsion conditions for which Eq. (98) predicts no axial shortening. When a cross-section is thin-walled and open, $J_\mathrm{t} \ll I_\mathrm{po}$ and Eq. (98) differs from conventional theory only by the ratio $1/(1+\mu)(1-2\mu)$. Of course one must also keep in mind that the derivation so far has been approximate and that equilibrium in the lateral directions has not been satisfied because t^{22} and t^{33} are of the order $(\phi_{,x})^2$.



$$\frac{I_{\bar{\omega}}}{A} = \tfrac{1}{4}\frac{\left(c^2 - d^2\right)^2}{c^2 + d^2}$$

$$\frac{I_{po}}{A} = \tfrac{1}{4}\left(c^2 + d^2\right)$$

$$\frac{J_t}{A} = \frac{c^2 d^2}{c^2 + d^2}$$

Fig. 7. Elliptic cross-section.

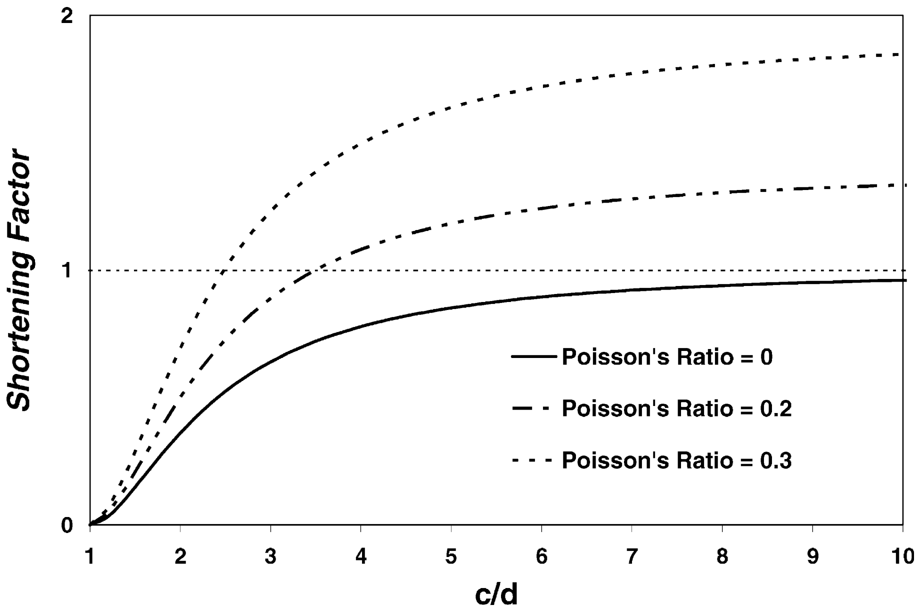


Fig. 8. Axial shortening factor for elliptic cross-section.

9. Conclusions

An endeavour has been made to review what is appropriate for the nonlinear analysis of beams. By postulating a strain energy density for an isotropic hyperelastic Hookean material, the constitutive relationships for the physical Lagrangian stresses on a beam cross-section were derived. The stress normal to the deformed surface is a function of the normal component of the longitudinal stretch while the shear is a function of the shear component of longitudinal stretch.

The buckling formula for a straight prismatic column including shear and axial deformations derived agreed with Haringx's formula. The problem of a straight prismatic three-dimensional Timoshenko-type beam with no warping was examined and elastica-type equations were derived. The example of pure torsion of a cylinder was also examined as the proposed formulation predicted no second-order axial shortening under pure torsion. This differs from conventional finite strain predictions where the axial shortening due to the Wagner effect is evident. When warping was included under pure torsion second-order axial shortening was predicted. A new formula for axial shortening under pure torsion was presented.

The last point that needs to be made is that although the differences between the formulas of Engesser and Haringx are negligible if the shear modulus is much greater than the initial axial stress, their differing approaches can lead to different nonlinear terms in other applications in structural analysis. Many stability analyses which involve second-order terms are based on a Hookean constitutive relationship between Green's strain tensor and the second Piola–Kirchhoff stress tensor. This approach has been placed into doubt.

Appendix A

The differential equation derived in Eq. (45) is used to estimate the buckling load for an initially straight prismatic column. The differential equation quoted in Eq. (45) is transformed by noting the following:

$$\begin{aligned}\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right) &= \frac{1}{r^2}P^*\sin\theta[P^*\cos\theta(1-m^*)-1]\\ \frac{\mathrm{d}\theta}{\mathrm{d}x}\left(\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right) &= \frac{1}{r^2}(P^*)^2(1-m^*)\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} - \frac{1}{r^2}P^*\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\\ \therefore \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 &= \frac{1}{r^2}(P^*)^2(1-m^*)\sin^2\theta - \frac{1}{r^2}4P^*\sin^2\frac{\theta}{2} + c\end{aligned} \tag{A.1}$$

with c being a constant of integration. The boundary condition at the left support is at $x = 0$, $\theta = \theta_0$ and $\mathrm{d}\theta/\mathrm{d}x = 0$ and therefore:

$$c = -\frac{1}{r^2}(P^*)^2(1-m^*)\sin^2\theta_0 + \frac{1}{r^2}4P^*\sin^2\frac{\theta_0}{2} \tag{A.2}$$

Substituting the equation for c into Eq. (A.1) gives:

$$\begin{aligned}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 &= \frac{1}{r^2}(P^*)^2(1-m^*)(\sin^2\theta - \sin^2\theta_0) - \frac{1}{r^2}4P^*\left(\sin^2\theta/2 - \sin^2\theta_0/2\right)\\ &= \frac{1}{r^2}(P^*)^2(1-m^*)(\cos^2\theta_0 - \cos^2\theta) - \frac{1}{r^2}2P^*(\cos\theta_0 - \cos\theta)\end{aligned} \tag{A.3}$$

Integrating, we can write the following:

$$\frac{L}{r} = \int_{\theta_0}^{-\theta_0} \frac{\mathrm{d}\theta}{\sqrt{(P^*)^2(1-m^*)(\cos^2\theta_0 - \cos^2\theta) - 2P^*(\cos\theta_0 - \cos\theta)}} \tag{A.4}$$

Here we introduce a new variable ϕ such that

$$\sin\frac{\theta_0}{2}\sin\phi = \sin\frac{\theta}{2} \tag{A.5}$$

with

$$\text{at} \quad x = 0, \quad \theta = \theta_0 \quad \phi = \frac{\pi}{2} \qquad \text{and at} \qquad x = L, \quad \theta = -\theta_0 \quad \phi = -\frac{\pi}{2} \tag{A.6}$$

and

$$d\theta = \frac{\sin\theta_0/2\cos\phi}{\frac{1}{2}\sqrt{1-\sin^2\theta_0/2\sin^2\phi}}\,d\phi \tag{A.7}$$

Eq. (A.4) is now transformed using the new variable. We also make the following assumption about the magnitude of the deflections.

$$\sin^2\frac{\theta_0}{2} \ll 1 \quad \text{and} \quad \sin^2\frac{\theta_0}{2}\sin^2\phi \ll 1 \tag{A.8}$$

Eq. (A.4) then reduces to the following

$$\frac{L}{r} = \int_{\pi/2}^{-\pi/2} \frac{d\phi}{\sqrt{(P^*)^2(m^*-1)+P^*}} \tag{A.9}$$

Integrating gives

$$\frac{L}{r} = \frac{\pi}{\sqrt{P^*(1+P^*(m^*-1))}} \tag{A.10}$$

Solutions to Eq. (A.10) do not exist if the denominator is zero. Therefore solutions to the following quadratic give an estimate of the buckling load.

$$\frac{\pi^2}{\left(\frac{L}{r}\right)^2} = P^*(1+P^*(m^*-1)) \tag{A.11}$$

References

Alwis, W.A.M., Wang, C.M., 1996. Wagner term in flexural–torsional buckling of thin-walled open-profile columns. Engineering Structures 18 (2), 125–132.

Attard, M.M., 1986. Nonlinear shortening and bending effect under pure torque of thin-walled open beams. Thin-Walled Structures 4, 165–177.

Attard, M.M., 2003. Finite Strain—Isotropic hyperelasticity. International Journal of Solids and Structures. In this issue.

Bazant, Z.P., 1971. Correlation study of formulations of incremental deformations and stability of continuous bodies. Journal of Applied Mechanics Transactions, ASME 38, 919–928.

Bazant, Z.P., Cedolin, L., 1991. Stability of Structures. Oxford University Press, Inc., New York.

Bazant, Z.P., 2003. Shear buckling of sandwich, fiber-composite and lattice columns, bearings and helical springs: paradox resolved. ASME Journal of Applied Mechanics 70, 75–83.

Engesser, F., 1889. Die Knickfestigkeit gerader Stäbe. Zeitschrift des Architekten und Ingenieur Vereins zu Hannover 35, 455.

Engesser, F., 1891. Die Knickfestigkeit gerader Stäbe. Zeutralblatt des Bauverwaltung 11, 483.

Gjelsvik, A., 1991. Stability of built-up columns. Journal of Engineering Mechanics 117 (6), 1331–1345.

Haringx, J.A., 1942. On the buckling and lateral rigidity of helical springs. Proceedings of the Koninklijke Nederlandse Akademie Van Wetenschappen 45, 533.

Kardomateas, G.A., Dancila, D.S., 1997. Buckling of moderately thick orthotropic columns: comparison of an elasticity solution with the Euler and Engesser/Haringx/Timoshenko formulae. International Journal of Solids and Structures 34 (3), 341–357.

Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity. Dover Publications, New York.

Ogden, R.W., 1997. Non-linear Elastic Deformations. Dover Publications, Inc., Mineola, New York.

Reissner, E., 1972. On one-dimensional finite-strain beam theory: the plane problem. Journal of Applied Mathematics and Physics 23, 795–804.

Reissner, E., 1982. Some remarks on the problem of column buckling. Ingenieur-Archiv 52, 115–119.

Simo, J.C., Hjelmstad, K.D., Taylor, R.L., 1984a. Numerical formulations of elasto-viscoplastic response of beams accounting for the effect of shear. Computer Methods in Applied Mechanics and Engineering 42, 301–330.

Simo, J.C., Kelly, J.M., 1984. The analysis of multilayer elastomeric bearings. Journal of Applied Mechanics 51 (June), 256–262.
Timoshenko, S.P., 1953. History of the Strength of Materials. McGraw-Hill, New York.
Timoshenko, S.P., Gere, J.M., 1963. Theory of Elastic Stability. McGraw-Hill International, London.
Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity. McGraw-Hill Kogakusha Ltd., Tokyo.
Zielger, H., 1982. Arguments for and against Engesser's buckling formulas. Ingenieur-Archiv 52, 105–113.